Phys 410
Fall 2015
Lecture #17 Summary
27 October, 2015

We continued to discuss the two-body conservative central force problem… We found that the orbit is described by $r(\varphi) = \frac{c}{1+\epsilon\cos\varphi}$, where $c = \frac{\ell^2}{\mu\gamma}$ is a length scale and ϵ is an un-determined positive dimensionless constant. This is the equation for the orbit of a planet around the sun, or a satellite around the earth. Note that when the un-determined constant $\epsilon > 1$, the denominator of $r(\varphi)$ has a zero for some angle φ, and the particle is off at infinity for that angle. This is an un-bounded orbit, like those with energy $E > 0$ noted in the last lecture. When $\epsilon < 1$ the values of $r(\varphi)$ are finite for all φ, and the orbit is bounded, like those with $E < 0$ noted above. The fact that $r(\varphi + 2\pi) = r(\varphi)$ means that the orbit is closed and periodic (this is not the case for other types of force interactions such as $F(r) \sim -1/r^3$).

The orbit for $\epsilon < 1$ is an ellipse and is described by $\frac{(x+d)^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a = \frac{c}{1-\epsilon^2}$ is the semi-major axis, $b = \frac{c}{\sqrt{1-\epsilon^2}}$ is the semi-minor axis, and $d = a\epsilon$ is the distance from the center of the ellipse to the focus (you will prove this in HW8). The ratio of semi-minor to semi-major axis lengths is $b/a = \sqrt{1-\epsilon^2}$, showing that ϵ is the ellipticity of the orbit. One can also derive Kepler's third law of planetary motion relating the orbital period τ and the semi-major axis as $\tau^2 = \frac{4\pi^2}{GM_{sun}} a^3$ for the case of a planet orbiting the sun (here one assumes that the mass of the planet is much smaller than that of the sun). Finally we calculated the total mechanical energy in the center of mass frame as $E = \frac{\gamma^2\mu}{2\ell^2}(\epsilon^2 - 1)$. This shows that orbits with $\epsilon > 1$ are un-bounded (and described by a hyperbola), and those with $\epsilon < 1$ are bounded. Orbits with $\epsilon = 1$ are parabolic. The un-bounded (hyperbolic) orbits have a range of angles φ for which there is no solution for $r(\varphi)$.

We next started a discussion of scattering theory. In the simplest scattering experiment one has a particle or entity (the projectile) that is launched with a known energy and momentum into a target, the projectile interacts with particles in the target, and then comes out as the same particle but with a new energy and momentum. More generally, the particle could be absorbed by the target, or be transformed into one or more different particles upon exiting the target. We can measure the exiting angle of the particle using spherical coordinates, with the z-axis along the initial projectile direction and the angular coordinates θ, φ specifying the new direction. Examples of scattering experiments include Rutherford scattering and angle-resolved photoemission spectroscopy (ARPES), which is basically the photoelectric effect on steroids.

The only quantity not controlled or measured in a typical scattering experiment is the impact parameter b of the projectile with respect to the target particle. The impact parameter is the distance of closest approach to the target particle, assuming no forces of interaction cause the projectile to change from it's initial direction. Because we cannot control the impact parameter, we have to perform many experiments in which all possible values of b are employed for the incident beam of projectiles. The objective of our calculations will be to find the functional relationship between the scattering angle and the impact parameter, namely $b = b(\theta)$, or $\theta = \theta(b)$.

Given the lack of control over the impact parameter, we resort to a statistical description of the resulting scattering. With such a description, we can write the number of particles scattered N_{scatt} in terms of the number of particles incident N_{inc} as $N_{scatt} = N_{inc} n_{target} \sigma$, where n_{target} is the density of target particles projected into the two-dimensional plane ($n_{target} \sim 1/m^2$) and σ is defined as the scattering cross section of each particle. σ is often measured in units of 'barns', which is $10^{-28} m^2$. We can generalize the concept of cross section to any process, including capture ($\sigma_{capture}$), ionization ($\sigma_{ionization}$), fission ($\sigma_{fission}$), etc. This is done by using the definition $N_{scatt,x} = N_{inc} n_{target} \sigma_x$ for process "x". These different processes are often referred to as "scattering channels."

Experiments start with a beam of projectile particles of identical structure and equal initial momenta and energy. The projectiles enter the target with all possible values of impact parameter. One then measures how many particles come out with angle of exit θ, φ and also the energy and momentum of the exiting particle. Our job is to identify the force of interaction between the projectile and target particles from the number of particles scattered through angle θ, φ, for all possible angles. We write the 'angle-resolved' scattering cross section as $N_{scatt}(into\ d\Omega\ around\ \theta, \varphi) = N_{inc} n_{target} \frac{d\sigma}{d\Omega}(\theta, \varphi) d\Omega$, where $\frac{d\sigma}{d\Omega}(\theta, \varphi)$ is called the differential scattering cross section (DSCS). Note that the element of differential solid angle is $d\Omega = \sin\theta\, d\theta d\varphi$. We expect that if this quantity is integrated over all possible exiting angles, we should recover the total scattering cross section for this process: $\sigma = \iint \frac{d\sigma}{d\Omega}(\theta, \varphi)\, d\Omega$. We shall assume that all scattering potentials are spherically symmetric, hence there will be no dependence on the φ coordinate.